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Uniform Approximation of Differentiable Functions by Algebraic Polynomials

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INTRODUCTION

In this paper we deduce a special form of Jackson's fundamental direct theorem of best approximation. We give an asymptotic best upper bound of the uniform approximation error of differentiable functions by algebraic polynomials.

We attempt to find the algebraic version of the theorem of Favard [1], and Achieser and Krein [2]:

$$E_n(W_r) = K_r(n+1)^{-r}. \tag{1}$$

In Sections 2 and 3 we prove the following

THEOREM 1. *Let $r \in \mathbb{N}$ and $n \geqslant r-1$. Then*

$$E_n(W_r) < K_r \cdot (n+1-r)!/(n+1)! \tag{2}$$

The constant K_r cannot be improved. This fact is a result of Bernstein's theorem [4]:

$$\lim_{n\to\infty} n^r \cdot E_n(W_r) = K_r. \tag{3}$$

The proofs of (1) and (3) can be found in [5].

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Bernstein's theorem also implies the existence of constants L_r such that for all $r \in \mathbb{N}$ and $n \geqslant r-1$,

$$E_n(W_r) \leqslant L_r \cdot n^{-r}; \tag{4}$$

but it does not give upper bounds for L_r.

1. Definitions

Let $C[-1, 1]$ be the space of all real continuous functions on $[-1, 1]$ with the norm

$$\| g\| := \max_{-1 \leqslant x \leqslant 1} |g(x)|.$$

Let $\mathscr{P}_n$ be the space of all algebraic polynomials of degree at most n; furthermore, let

$$E_n(g) := \inf_{p \in \mathscr{P}_n} \| g - p\|$$

be the approximation error on $C[-1, 1]$. Let W_r be the space of all functions g with g, g', g'',..., $g^{(r-1)}$ in $C[-1, 1]$ and $|g^{(r)}| \leqslant 1$ a.e. The constant $E_n(W_r)$ is defined by

$$E_n(W_r) := \sup_{g \in W_r} E_n(g).$$

Analogously let $C_{2\pi}$ be the space of all real continuous 2π-periodic functions with the norm ($K := [\emptyset, 2\pi]$)

$$\|f\|^* := \max_{x \in K} |f(x)|.$$

Let $\mathscr{T}_n$ be the space of all trigonometric polynomials of degree at most n and let

$$E_n^*(f) := \inf_{t \in \mathscr{T}_n} \|f - t\|^*$$

be the approximation error on $C_{2\pi}$. Let W_r^* be the space of all functions f with f, f', f'',..., $f^{(r-1)}$ in $C_{2\pi}$ and $|f(r)| \leqslant 1$ a.e. The constant $E_n^*(W_r^*)$ is defined by

$$E_n^*(W_r^*) := \sup_{f \in W_r^*} E_n^*(f).$$

According to Favard [1], and Achieser and Krein [2], we define

$$K_r := (4/\pi) \sum_{m=0}^{\infty} (-1)^{m(r+1)}(2m+1)^{-r-1}. \tag{5}$$

This implies the inequality

$$K_2 < K_4 < \cdots < 4/\pi < \cdots < K_3 < K_1 = \pi/2. \tag{6}$$

2. Turning the Algebraic into a Trigonometric Problem

The purpose of this section is to replace $E_n(g)$ by $E_n^*(g \circ \sin)$, and to deduce upper bounds for $E_n(W_1)$ and $E_n(W_2)$.

THEOREM 2. *Let $r \in \mathbb{N}$, $n \geqslant r-1$, $g \in W_r$, and let*

$$f(t) := \int_0^{\sin t} g^{(r)}(u)(\sin t - u)^{r-1}\, du/(r-1)! \tag{7}$$

Then

$$E_n(g) = E_n^*(f).$$

Proof. Let p be the best approximation of g in $\mathscr{P}_n$. Then $g-p$ has at least $n+2$ alternation points, and $(g-p) \circ \sin$ has at least $2n+2$ alternation points in $[0, 2\pi)$. Therefore $p \circ \sin$ is the best approximation of $g \circ \sin$, and

$$E_n(g) = \| g-p\| = \| g \circ \sin - p \circ \sin\|^* = E_n^*(g \circ \sin). \tag{8}$$

By Taylor's theorem we have

$$g(\sin t) = \sum_{k=0}^{r-1} g^{(k)}(0) \cdot (\sin t)^k/k! + f(t). \tag{9}$$

Since $n \geqslant r-1$, we have $E_n^*(g \circ \sin) = E_n^*(f)$. ∎

THEOREM 3. *Let $r \in \mathbb{N}_0$, $f \in C_{2\pi}^r$, $n \in \mathbb{N}_0$, and let $\omega(h)$ be a concave modulus of continuity of $f^{(r)}$. Then*

$$E_n{}^*(f) \leqslant \frac{1}{2} K_r (n+1)^{-r}\, \omega\left(\frac{\pi}{n+1}\right). \tag{10}$$

Proof. See [3].

To prove Theorem 1 for $r=1,2$ we need some special considerations listed in the following theorems:

THEOREM 4. *Let* $n \in \mathbb{N}_0$. *Then*

$$E_n(W_1) \leqslant \sin(\pi/2(n+1)) < K_1/(n+1). \tag{11}$$

Proof. Let $g \in W_1$, $0 \leqslant h \leqslant \pi$, a, $b \in \mathbb{R}$, $0 \leqslant b-a \leqslant h$; for f see (7). $\omega(h) := 2\sin(h/2)$ is a concave modulus of continuity of f, as

$$|f(b)-f(a)| \leqslant |\sin b - \sin a| \leqslant 2\sin(h/2).$$

By Theorem 3 we get

$$E_n(g) = E_n^*(f) \leqslant \tfrac{1}{2}\omega(\pi/(n+1)) = \sin(\pi/2(n+1)). \quad \blacksquare$$

THEOREM 5. *Let* $n \in \mathbb{N}$. *Then*

$$E_n(W_2) \leqslant K_2(n+1)^{-2} < K_2(n-1)!/(n+1)!. \tag{12}$$

Proof. Let $g \in W_r$, $t \in \mathbb{R}$; for f see (7). Then

$$\begin{aligned} |f''(t)| &= \left|(\cos t)^2\, g''(\sin t) - \sin t. \int_0^{\sin t} g''(u)\,du\right| \\ &\leqslant (\cos t)^2 + (\sin t)^2 = 1. \end{aligned} \tag{13}$$

Therefore we obtain that $f \in W_2^*$, and by (1) and Theorem 2 we have

$$E_n(g) = E_n^*(f) \leqslant K_2(n+1)^{-2}. \quad \blacksquare$$

3. AN UPPER BOUND OF $E_n(W_r)$

To prove Theorem 1 for $r \geqslant 3$, we need several definitions and lemmas. For all $r \geqslant 3$ and $j \in \mathbb{N}_0$ let

$$p_{rj}(t) := ((d/dt)^{r+j-1}(\sin t - u)^{r-1})_{u=\sin t}/(r-1)!, \tag{14}$$

and

$$B_{rj} := \|\cos \cdot p_{rj}\|^*. \tag{15}$$

It is obvious that $p_{rj} \in \mathscr{E}_{r-1}$.

LEMMA 6. *Let $r \geqslant 3$, $n \geqslant r-1$ and $g \in W_r$. Then*

$$E_n(g) \leqslant \sum_{j=0}^{\infty} B_{rj} K_{r+j} (n+1)^{-r-j}. \tag{16}$$

Proof. Using Theorem 2 we only have to prove the inequality

$$E_n^*(f) \leqslant \sum_{j=0}^{\infty} B_{rj} K_{r+j} (n+1)^{-r-j}. \tag{17}$$

We split up f into several functions f_j so that $B_{rj}^{-1} \cdot f_j \in W_{r+j}^*$. For all j and $s \in \mathbb{N}_0$ we define f_j and $\tilde{f}_s$ by

$$\begin{gathered} f_j \in C_{2\pi}, \qquad \int_K f_j = 0, \\ f_j^{(r+j)}(t) = g^{(r)}(\sin t) \cdot \cos t \cdot p_{rj}(t) + \text{const.}, \end{gathered} \tag{18}$$

and

$$\begin{aligned} &\tilde{f}_s \in C_{2\pi}, \qquad \int_K \tilde{f}_s = \int_K f, \\ &\tilde{f}_s^{(r+s)}(t) = ((r-1)!)^{-1} \int_0^{\sin t} g^{(r)}(\sin t) \left(\frac{d}{dt}\right)^{r+s} \\ &\qquad \cdot (\sin t - u)^{r-1}\, du + \text{const.} \end{aligned} \tag{19}$$

By calculation of $f(r)$ we get

$$f^{(r)} = f_0^{(r)} + \tilde{f}_0^{(r)} + \text{const.} \tag{20}$$

Since $\int_K f^{(r)} = 0 = \int_K f_0^{(r)} + \int_K \tilde{f}_0^{(r)}$ and $\int_K f = \int_K f_0 + \int_K \tilde{f}_0$ we obtain $f = f_0 + \tilde{f}_0$. We also have

$$\tilde{f}_{s-1}^{(r+s)} = (d/dt)\, \tilde{f}_{s-1}^{(r+s-1)} = f_s^{(r+s)} + \tilde{f}_s^{(r+s)} + \text{const.} \tag{21}$$

By similar arguments as above we obtain $\tilde{f}_{s-1} = f_s + \tilde{f}_s$, and thus for all $s \in \mathbb{N}_0$

$$E_n^*(f) = E_n^* \left(\sum_{j=0}^{s} f_j + \tilde{f}_s \right) \leqslant \sum_{j=0}^{\infty} E_n^*(f_j) + E_n{}^*(\tilde{f}_s). \tag{22}$$

Using Theorem 3 and

$$|f_j^{(r+j)}(t) - f_j^{(r+j)}(t')| \leqslant 2\, \|\cos \cdot p_{rj}\|^* = 2B_{rj}$$

we obtain

$$E_n^*(f_j) \leqslant B_{rj} K_{r+j} (n+1)^{-r-j}. \tag{23}$$

Since

$$\begin{aligned} &|\tilde{f}_s^{(r+s)}(t) - \tilde{f}_s^{(r+s)}(t')| \\ &\quad \leqslant 2 \int_0^1 \left\| \left(\frac{d}{dt}\right)^{r+s} (\sin t - u)^{r-1} \right\|^* du/(r-1)! \\ &\quad \leqslant 2 \int_0^1 (r-1)^{r+s} (1+u)^{r-1} \, du/(r-1)! \\ &\quad < 2^{r+1} (r-1)^{r+s}/r!, \end{aligned} \tag{24}$$

we have

$$\lim_{s\to\infty} E_n^*(\tilde{f}_s) \leqslant \lim_{s\to\infty} (2^r/r!) \, K_{r+s}((r-1)/(n+1))^{r+s} = 0. \tag{25}$$

Combining (22), (23) and (25) we finally get (17). ∎

LEMMA 7. *Let $m \geqslant r \geqslant 3$. Then*

$$\sum_{j=0}^{\infty} B_{rj} m^{-r-j} \leqslant (1 - m^{-1}) \cdot (m-r)!/m!. \tag{26}$$

Proof. For $r = 3$ we have

$$B_{3j} = \| \mathrm{cost}\, t \cdot ((\cos 2t)^{(j)} + \sin t \cdot (\sin t)^{(j)}) \|^* \leqslant 2^j$$

and

$$\begin{aligned} \sum_{j=0}^{\infty} B_{3j} m^{-3-j} &\leqslant m^{-3} \sum_{j=0}^{\infty} (2/m)^j \\ &= (1 - m^{-1}) \cdot (m-3)!/m!. \end{aligned} \tag{27}$$

Now suppose (26) being proved for r. We prove (26) for $r+1$: For $j \geqslant 1$ we get $p_{r+1,j} = \cos \cdot p_{rj} + p'_{r+1,j-1}$. Since $p_{r+1,0} = \cos^r = \cos \cdot p_{r0}$, we obtain

$$p_{r+1,j} = \sum_{k=0}^{j} (\cos \cdot p_{r,j-k})^{(k)}. \tag{28}$$

Using Bernstein's inequality we have

$$B_{r+1,j} \leqslant \| p_{r+1,j} \|^* \leqslant \sum_{k=0}^{j} r^k B_{r,j-k}, \tag{29}$$

and therefore

$$\sum_{j=0}^{\infty} B_{r+1,j} m^{-r-1-j}$$
$$\leqslant \sum_{j=0}^{\infty} B_{rj} m^{-r-j} \cdot m^{-1} \cdot \sum_{k=0}^{\infty} (r/m)^k$$
$$= (m-r)^{-1} \cdot \sum_{j=0}^{\infty} B_{rj} m^{-r-j} \leqslant (1-m^{-1}) \cdot (m-r-1)!/m!. \quad \blacksquare$$

LEMMA 8. *Let $m \geqslant r \geqslant 3$. Then*

$$\sum_{j=0}^{\infty} B_{rj} K_{r+j} m^{-r-j} < K_r (m-r)!/m!. \tag{30}$$

Proof. For r odd we have $K_r \geqslant K_{r+j}$ (see (6)), and by Lemma 7 we get

$$\sum_{j=0}^{\infty} B_{rj} K_{r+j} m^{-r-j} \leqslant (1-m^{-1}) \cdot K_r \cdot (m-r)!/m!. \tag{31}$$

For r even we have $K_{r+1} \geqslant K_{r+j}$, and thus

$$\sum_{j=0}^{\infty} B_{rj} K_{r+j} m^{-r-j}$$
$$\leqslant K_r \sum_{j=0}^{\infty} B_{rj} m^{-r-j} + (K_{r+1} - K_r) \sum_{j=1}^{\infty} B_{rj} m^{-r-j}$$
$$\leqslant K_r (1-m^{-1})(m-r)!/m! + (K_{r+1} - K_r)((m-r)!/m! - m^{-r})$$
$$= (1 - m^{-1} + (K_{r+1}/K_r - 1)(1 - m^{-r} m!/(m-r)!))\, K_r (m-r)!/m!. \tag{32}$$

Now we use two simple estimates, namely,

$$1 - m^{-r} \cdot m!/(m-r)! < 1 - m^{-r}(m-r)^r = 1 - (1 - r/m)^r < r^2/m, \tag{33}$$

and

$$K_{r+1}/K_r = \left(\sum_{m=0}^{\infty} (1+2m)^{-r-2} \right) \Big/ \left(\sum_{m=0}^{\infty} (-1)^m (1+2m)^{-r-1} \right)$$
$$< \left(1 + \sum_{m=1}^{\infty} 3^{-r} (1+2m)^{-2} \right) \Big/ (1 - 3^{-r-1})$$
$$= (1 + (\pi^2/8 - 1)\, 3^{-r})/(1 - 3^{-r-1})$$
$$= 1 + (\pi^2/8 - 1 + 3^{-1})/(3^r - 3^{-1})$$
$$< 1 + 1/((1+2)^r - 1)$$
$$< 1 + 1/(1 + 2r + 2r(r-1) - 1) = 1 + 1/2r^2. \tag{34}$$

Combining (32), (33) and (34) we obtain

$$\sum_{j=0}^{\infty} B_{rj} K_{r+j} m^{-r-j} < (1 - m^{-1} + 2^{-1} m^{-1}) K_r (m-r)!/m! \quad \blacksquare \tag{35}$$

Now Theorem 1 follows by Lemma 6 and Lemma 8 for $r \geqslant 3$, and by Theorem 4, resp. 5 for $r = 1$, resp. 2.

Remarks

Another representation of Theorem 1 is the following.

COROLLARY 9. *Let* $r \in \mathbb{N}$, $n \geqslant r-1$, $g \in C^r[-1, 1]$. *Then there exists a polynomial* $p \in \mathscr{P}_n$ *with*

$$\| g - p \| \leqslant \| g^{(r)} \| \cdot K_r \cdot (n + 1 - r)!/(n + 1)!. \tag{36}$$

Proof. If $\| g^{(r)} \| = 0$, then $g \in \mathscr{P}_{r-1} \subset \mathscr{P}_n$. If $\| g^{(r)} \| \neq 0$, (36) follows by Theorem 1 and $g/\| g^{(r)} \| \in W_r$. $\blacksquare$

Note that for small n the upper bound of $E_n(W_r)$ given is not very good. For $n = r - 1$ it is known (see Fisher [5]) that

$$E_{r-1}(W_r) = 2^{1-r}/r!,$$

while our inequality gives only

$$E_{r-1}(W_r) < K_r/r!.$$

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